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# On the compatibility of Lorentz metrics with linear connections on four-dimensional manifolds

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**Abstract**
This paper considers four-dimensional manifolds upon which there is a Lorentz metric $h$ and a symmetric connection $\Gamma$ which are originally assumed unrelated. It then derives sufficient conditions on $h$ and $\Gamma$ (expressed through the curvature tensor of $\Gamma$) for $\Gamma$ to be the Levi-Civita connection of some (local) Lorentz metric $g$ and calculates the relationship between $g$ and $h$. Some examples are provided which help to assess the strength of the sufficient conditions derived.

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## 1. Introduction

Let $M$ be a connected $n$-dimensional manifold admitting a smooth metric $g$ of arbitrary signature, associated Levi-Civita connection $\Gamma$ and corresponding curvature tensor $R$. Then, if $\nabla$ denotes the covariant derivative from $\Gamma$, one has $\nabla g = 0$ and, in components in any coordinate system,

$$g_{ae}R^e{}_{bcd} + g_{be}R^e{}_{acd} = 0, \qquad g_{ae}R^e{}_{bcd;f} + g_{be}R^e{}_{acd;f} = 0, \ldots, \tag{1}$$

where a semicolon denotes the covariant derivative $\nabla$ in component form. This has led to the following question (or variants of it) in the literature. Suppose a connected $n$-dimensional manifold $M$ admits a smooth symmetric linear connection $\Gamma$ with the corresponding curvature tensor $R$ and suppose, in addition, that $M$ admits a global smooth metric $h$ of arbitrary signature such that on $M$

$$h_{ae}R^e{}_{bcd} + h_{be}R^e{}_{acd} = 0 \tag{2}$$

$$h_{ae}R^e{}_{bcd;f_1} + h_{be}R^e{}_{acd;f_1} = 0 \tag{3.1}$$

$$\vdots \qquad\qquad \vdots$$

$$h_{ae}R^e{}_{bcd;f_1\ldots f_k} + h_{be}R^e{}_{acd;f_1\ldots f_k} = 0 \tag{3.k}$$

  

for some integer $k$. Under which conditions is $\Gamma$ a *metric (or locally metric) connection*? That is, under which conditions does there exist a global metric $g$ on $M$, of arbitrary signature, whose Levi-Civita connection is $\Gamma$ (or, given $m \in M$, does there exist an open neighbourhood $U$ of $m$ and metric $g$ of arbitrary signature on $U$ whose Levi-Civita connection is the restriction to $U$ of $\Gamma$). If so, how are $g$ and $h$ related?

If one is somehow able to find the holonomy group of $\Gamma$, then the problem is partly solved because then $\Gamma$ is a metric connection if and only if for some $m \in M$ there is a non-degenerate quadratic form on the tangent space $T_mM$ to $M$ at $m$, of signature $(p, q)$ for non-negative integers $p, q$ with $p+q = n$, which is preserved by the holonomy group associated with $\Gamma$. If such is the case, $\Gamma$ is compatible with a metric $g$ on $M$ of signature $(p, q)$, that is $\nabla g = 0$, and the holonomy group of $\Gamma$ is a Lie subgroup of $O(p, q)$ [1, 2]. In the case $n = 4$ and if $\Gamma$ is fixed and assumed to be the Levi-Civita connection of a Lorentz metric on $M$, the holonomy group of $\Gamma$ can be used to find *all* metrics on $M$ compatible with $\Gamma$ and these may (depending on the holonomy group) have any of the signatures $(1, 3)$, $(0, 4)$ or $(2, 2)$ [3]. Remaining in the $n = 4$ case and with the curvature tensor $R$ fixed and assumed to arise from a Levi-Civita connection $\Gamma$ compatible with a Lorentz metric on $M$, it is known that this metric is generically determined up to a constant conformal factor and hence, generically, $\Gamma$ is uniquely determined [4–7]. It is also known [8] that on a large class of four-dimensional manifolds a symmetric linear connection $\Gamma$ satisfying (2) for some metric $h$ on $M$ is necessarily *locally* metric (in the sense that each $m \in M$ admits an open neighbourhood $U$ such that the restriction of $\Gamma$ to $U$ is metric). However, the procedure involves the investigation of a $36 \times 36$ matrix and is geometrically obscure. In this paper, the question raised at the beginning of this section for the case $n = 4$ and with $h$ of Lorentz signature will be considered, that is, if $M$ admits a symmetric linear connection $\Gamma$, associated curvature $R$ and metric tensor $h$ of Lorentz signature (*not* assumed related to $\Gamma$ or $R$ in any way) such that (2) and (3.1)–(3.$k$) hold for some $k$, is $\Gamma$ a metric (or locally metric) connection on $M$ and, if so, how is $h$ related to the metric or metrics compatible with $\Gamma$?

A standard notation will be used, with round and square brackets denoting the usual symmetrization and skew-symmetrization of indices, respectively, and a comma denotes a partial derivative. A skew-symmetric tensor $F$ of type $(0, 2)$ or $(2, 0)$ at $m$ is called a *bivector*. If $F(\neq 0)$ is such a bivector, the rank of any of its (component) matrices is either 2 or 4. In the former case, one may write (e.g. in the $(2, 0)$ case) $F^{ab} = 2r^{[a}s^{b]}$ for $r, s \in T_mM$ (or, alternatively, $F = r \wedge s$) and $F$ is called *simple*, with the two-dimensional subspace (2-space) of $T_mM$ spanned by $r, s$ referred to as the *blade* of $F$. In the latter case, $F$ is called *non-simple*.

The metric $h(m)$ converts $T_mM$ into a Lorentz inner product space and thus it makes sense to refer to vectors in $T_mM$ and covectors in the cotangent space $T^*_mM$ to $M$ at $m$ (using $h(m)$ to give a unique isomorphism $T_mM \leftrightarrow T^*_mM$, that is, to raise and lower tensor indices) as being *timelike*, *spacelike*, *null* or *orthogonal*, using the signature $(-, +, +, +)$. The same applies to one-dimensional subspaces (*directions*) and two- and three-dimensional subspaces of $T_mM$ or $T^*_mM$. A *simple* bivector at $m$ is then called *timelike* (respectively, *spacelike* or *null*) if its blade at $m$ is a timelike (respectively a *spacelike* or *null*) 2-space at $m$. A *non-simple* bivector $F$ at $m$ may, by a standard argument ([9], see also [4]), be shown to uniquely determine an orthogonal pair of 2-spaces at $m$, one spacelike and one timelike, and which are referred to as the *canonical pair of blades* of $F$. A tetrad $(l, n, x, y)$ of members of $T_mM$ is called a null tetrad at $m$ if the only non-vanishing inner products between its members at $m$ are $h(l, n) = h(x, x) = h(y, y) = 1$. Thus $l$ and $n$ are null.

It is also remarked that the tensor, with local components $R_{abcd} \equiv h_{ae}R^e{}_{bcd}$, satisfies (using (2)) $R_{abcd} = -R_{bacd} = -R_{abdc}$ and $R_{abcd} + R_{adbc} + R_{acdb} = 0$. It then follows after some index juggling (see, e.g., [10]) that $R_{abcd} = R_{cdab}$.

## 2. Preliminary results

Let $M$ be a four-dimensional smooth connected Hausdorff manifold admitting a smooth symmetric linear connection $\Gamma$ with associated curvature tensor $R$ and a global smooth Lorentz metric $h$ such that (2) holds. No relation between $h$ and $\Gamma$ is assumed other than (2). It will be convenient to describe a simple algebraic classification of $R$ (relative to $h$ —but see the remark following (11)). This classification is easily described geometrically and is a pointwise classification. It has been described before in a more specific context (see, e.g., [4, 11]) but since its use here is slightly different, it will be briefly described.

Define a linear map $f$ from the six-dimensional vector space of type $(2, 0)$ bivectors at $m$ into the vector space of type $(1, 1)$ tensors at $m$ by $f : F^{ab} \to R^a{}_{bcd}F^{cd}$. Condition (2) shows that if a tensor $T$ is in the range of $f$, then

$$h_{ae}T^e{}_b + h_{be}T^e{}_a = 0 \qquad \left(\Rightarrow T_{ab} = -T_{ba}, T_{ab} = h_{ae}T^e{}_b\right) \tag{4}$$

and so $T$ can be regarded as a member of the matrix representation of the Lie algebra of the pseudo-orthogonal (Lorentz) group of $h(m)$. Using $f$ one can divide the curvature tensor $R(m)$ into five classes.

*Class A*. This is the most general curvature class and the curvature will be said to be of (curvature) class $A$ at $m \in M$ if it is not in any of classes $B$, $C$, $D$ or $O$ below.

*Class B*. The curvature tensor is said to be of (curvature) class $B$ at $m \in M$ if the range of $f$ is two-dimensional and consists of all linear combinations of type $(1, 1)$ tensors $F$ and $G$, where $F^a{}_b = x^a y_b - y^a x_b$ and $G^a{}_b = l^a n_b - n^a l_b$ with $l, n, x, y$ a null tetrad at $m$. The curvature tensor at $m$ can then be written as

$$R_{abcd} \equiv h_{ae}R^e{}_{bcd} = \frac{\alpha}{2}F_{ab}F_{cd} - \frac{\beta}{2}G_{ab}G_{cd}, \tag{5}$$

where $\alpha, \beta \in \mathbf{R}, \alpha \neq 0 \neq \beta$. The symmetrized cross term in $F$ and $G$ vanishes because $R_{a[bcd]} = 0$. This class can be further split into classes $B_1$ and $B_2$, where $B_1$ (respectively, $B_2$) applies if $\alpha \neq \beta$ (respectively, $\alpha = \beta$).

*Class C*. The curvature tensor is said to be of (curvature) class $C$ at $m \in M$ if the range of $f$ is two- or three-dimensional and if there exists $0 \neq k \in T_m M$ such that each of the type $(1, 1)$ tensors in the range of $f$ contains $k$ in its kernel (i.e. each of their matrix representations $F$ satisfies $F^a{}_b k^b = 0$).

*Class D*. The curvature tensor is said to be of (curvature) class $D$ at $m \in M$ if the range of $f$ is one-dimensional. It follows that the curvature components satisfy $R_{abcd} = \lambda F_{ab}F_{cd}$ at $m (0 \neq \lambda \in \mathbf{R})$ for some bivector $F$ at $m$ which then satisfies $F_{a[b}F_{cd]}$ and is thus simple. It also follows that there exist two independent members $r, s \in T_m M$ such that $F_{ab}r^b = F_{ab}s^b = 0$ and hence that $r$ and $s$ lie in the kernel of each tensor in the range of $f$.

*Class O*. The curvature tensor is said to be of (curvature) class $O$ at $m \in M$ if it vanishes at $m$.

The following remarks can be checked in a straightforward manner [4].

(i) For classes $A$ and $B$ there does not exist $0 \neq k \in T_m M$ such that $F^a{}_b k^b = 0$ for *every* $F$ in the range of $f$.
(ii) For class $A$, the range of $f$ has dimension at least 2 and if this dimension is 4 or more the class is necessarily $A$.
(iii) The vector $k$ in the definition of class $C$ is unique up to a scaling.
(iv) For classes $A$ and $B$ there does not exist $0 \neq k \in T_m M$ such that $R^a{}_{bcd}k^d = 0$, whereas this equation has exactly one independent solution for class $C$ and two for class $D$.

(v) The five classes $A$, $B$, $C$, $D$ and $O$ are mutually exclusive and exhaustive for the curvature tensor at $m$. If the curvature class is the same at each $m \in M$, then $M$ will be said to be of that class.

Denoting by the same symbols $A$, $B$ (or $B_1$ and $B_2$), $C$ , $D$ and $O$ those subsets of $M$ consisting of precisely those points where the curvature tensor has that class, one may decompose $M$ disjointly as $M = A \cup B_1 \cup B_2 \cup C \cup D \cup O$ and then, disjointly as

$$M = A \cup \operatorname{int} B_1 \cup \operatorname{int} B_2 \cup \operatorname{int} C \cup \operatorname{int} D \cup \operatorname{int} O \cup Z, \tag{6}$$

where int denotes the manifold topology interior operator and $Z$ is the (necessarily closed) subset of $M$ defined by the disjointness. The idea is to show that $Z$ has empty interior, $\operatorname{int} Z = \emptyset$. A similar decomposition was considered in [4, 11, 12], but in those works $B$ was not subdivided into $B_1$ and $B_2$ and so the 'leftovers' subset $Z$ may differ from that in (6) since $\operatorname{int} B_1 \cup \operatorname{int} B_2$ may be a proper subset of $\operatorname{int} B$. To show that $\operatorname{int} Z = \emptyset$ in (6) it is first noted that $A$ is open in $M$ (i.e. $\operatorname{int} A = A$) and that $A \cup B$ is also open in $M$ [4]. So let $U \subseteq Z$ be open in $M$. Then by the disjointness of the decomposition (6) $U \cap A = \emptyset$ and so $U \cap (A \cup B) = U \cap B \equiv W$ and $W$ is open in $M$. The disjointness of (6) also shows that, if $W \neq \emptyset$, $W$ must intersect each of $B_1$ and $B_2$ non-trivially, so let $m \in W \cap B_1$ and consider the linear map $f'$ from the vector space of type (2, 0) bivectors at $m$ into itself given by $f' : F^{ab} \to R^{ab}{}_{cd} F^{cd}$. The characteristic polynomial of this map has, from (5), three distinct solutions $\alpha$, $\beta$ and 0 at $m$ and since the first two of these are *simple* roots they give rise to smooth functions $\tilde{\alpha}$ and $\tilde{\beta}$ defined on some neighbourhood $V$ of $m$ (and with $\tilde{\alpha}(m) = \alpha$ and $\tilde{\beta}(m) = \beta$) which are solutions of the characteristic polynomial of $f'$ and distinct from each other and from zero in $V$ [13]. It follows that $m \in V \cap W \subseteq B_1$ with $V \cap W$ open and disjoint from $B_2$ (since at points of $B_2$ the characteristic polynomial of $f'$ has only two distinct solutions $\alpha (= \beta)$ and 0). Thus $V \cap W \subseteq \operatorname{int} B_1$ and hence $U \cap \operatorname{int} B_1 \neq \emptyset$, contradicting the disjointness of (6). It follows that $W$ and hence $U \cap B$ are empty. From here the argument follows [4] to get $U = \emptyset$ and hence $\operatorname{int} Z = \emptyset$. The following is thus established.

**Theorem 1.** *In the disjoint decomposition (6), $Z$ is closed,* $\operatorname{int} Z = \emptyset$ *and so $M \backslash Z$ is an open dense subset of $M$.*

It is remarked that the reason for decomposing $B$ as $B_1 \cup B_2$ is that the above argument leading to the local smooth functions $\tilde{\alpha}$ and $\tilde{\beta}$ and which extend $\alpha$ and $\beta$ does not apply if $\alpha = \beta$ at $m$ unless $m \in \operatorname{int} B_2$ (i.e. the problem lies with the points in $B_2 \setminus \operatorname{int} B_2$).

It is also remarked that in the open subset $\operatorname{int} C$ of $M$ the non-zero tangent vectors $k$ introduced in the definition of this type give rise to a *smooth* integrable one-dimensional distribution on $\operatorname{int} C$. To see this one recalls that the curvature tensor satisfies $R^a_{bcd} k^d = 0$ at each $m \in \operatorname{int} C$ and that $k$ is unique (up to a scaling) in satisfying this equation. The smoothness of the distribution defined follows from the smoothness of the curvature in the following way [4, 14]. Let $m \in \operatorname{int} C$ and let $0 \neq k \in T_m M$ satisfy the above equation. Choose a coordinate domain $U \subseteq \operatorname{int} C$ about $m$ in which $k^4 = 1$ at $m$. The conditions $R^a_{bcd} k^d = 0$ on $U$ then reduce to a system of equations of the form $\sum_{\alpha=1}^{3} E_{p\alpha} k^\alpha = E_p$ for functions $E_{p\alpha}$ and $E_p : U \to \mathbf{R}$ and $p = 1, 2, 3$. Since the linear system so formed has rank equal to 3 at $m$ (since there $k^\alpha$ are uniquely determined), then by taking $k^4 = 1$ on $U$ one obtains a linear system for $k^\alpha$ of rank 3 over $U$. Cramer's rule then reveals smooth solutions for the components of $k^\alpha$ and hence a local nowhere zero vector field $k$ satisfying $R^a{}_{bcd} k^d = 0$.

In the open subset $\operatorname{int} D$ it can be shown that for $m \in \operatorname{int} D$ there exists an open coordinate neighbourhood $U \subseteq \operatorname{int} D$ of $m$ on which the curvature tensor satisfies $R_{abcd} = \lambda F_{ab} F_{cd}$, where $F$ is a smooth bivector and $\lambda$ a smooth real-valued function on $U$. A similar argument to the previous one shows that $U$ may be chosen so that there are smooth vector fields $r$ and $s$ on $U$ such that at each $m' \in U$, $r(m')$ and $s(m')$ are independent and satisfy $R^a{}_{bcd}r^d = R^a{}_{bcd}s^d = 0$ [13].

If $m \in \operatorname{int} B_1$, the curvature tensor takes the form (5) over some coordinate neighbourhood $U \subseteq \operatorname{int} B_1$ of $m$ with $\alpha, \beta$ smooth nowhere zero, nowhere equal real-valued functions and $F$ and $G$ smooth bivectors on $U$. Similar remarks apply if $m \in \operatorname{int} B_2$ with $\alpha = \beta$ on $U \subseteq \operatorname{int} B_2$. For the latter two ($B_1$ and $B_2$) cases, a null tetrad $l, n, x, y$ of smooth vector fields exists on $U$ satisfying (5) on $U$.

## 3. The main results

Let $M$, $\Gamma$, $R$ and $h$ be as described at the beginning of section 2 and suppose that (2) holds. If (3.1) to (3.$k$) also hold, then it follows that on $M$

$$h_{ae;f_1}R^e{}_{bcd} + h_{be;f_1}R^e{}_{acd} = 0 \tag{7.1}$$

$$\vdots \qquad\qquad \vdots$$

$$h_{ae;f_1\ldots f_k}R^e{}_{bcd} + h_{be;f_1\ldots f_k}R^e{}_{acd} = 0 \tag{7.$k$}$$

and conversely the set of equations (2) and (7.1) to (7.$k$) imply the set (2) and (3.1) to (3.$k$). Thus conditions (2) and (3.1) to (3.$k$) are equivalent to conditions (2) and (7.1) to (7.$k$). The aim is to use this equivalence to show when the original conditions (2) and (3.1) to (3.$k$) for a particular $k$ imply the existence of a local or global Lorentz metric $g$ on $M$ compatible with $\Gamma$ and to display the relationship between $g$, $h$ and the geometry of $R$ as expressed through its curvature type. The method to be used employs a theorem in [4, 15] and involves the following idea. Let $m \in M$ and $T$ be a type $(0, a)$ tensor at $m$ with $a \geqslant 2$ and with $T$ symmetric in its first two indices (so $T$ has components $T_{abc\ldots d}$ and $T_{abc\ldots d} = T_{bac\ldots d}$). Suppose also that at $m$ (cf (2))

$$T_{aeg\ldots h}R^e{}_{bcd} + T_{beg\ldots h}R^e{}_{acd} = 0. \tag{8}$$

Then for any tensor $S$ of type $(a-2, 0)$, the tensor $h'_{ab} = T_{abc\ldots d}S^{c\ldots d}$ is symmetric and satisfies (2). This means that for any $F$ in the range of the map $f$ at $m$ described earlier

$$h'_{ae}F^e{}_b + h'_{be}F^e{}_a = 0. \tag{9}$$

Each such $F$ in the range of $f$ imposes strong algebraic constraints on $h'$, these constraints being conveniently written down for each of the curvature classes for $R$ at $m$ in terms of the original Lorentz metric $h$ and the features of the particular curvature class. They are, in the notation of section 2 [4, 15],

Class A $\quad h'_{ab} = \alpha h_{ab}$ (10$a$)

Class B $\quad h'_{ab} = \alpha h_{ab} + 2\beta l_{(a}n_{b)} = (\alpha+\beta)h_{ab} - \beta(x_ax_b + y_ay_b)$ (10$b$)

Class C $\quad h'_{ab} = \alpha h_{ab} + \beta k_ak_b$ (10$c$)

Class D $\quad h'_{ab} = \alpha h_{ab} + \beta r_ar_b + \gamma s_as_b + 2\delta r_{(a}s_{b)},$ (10$d$)

where $\alpha, \beta, \gamma, \delta \in \mathbf{R}$ and the completeness relation $h_{ab} = 2l_{(a}n_{b)} + x_ax_b + y_ay_b$ was used in (10$b$). These expressions allow, depending on the curvature class, expressions for the covariant derivatives of $h$ to be written out reasonably conveniently since, from (2) and (7.1) to (7.$k$), they satisfy the conditions asked for in (8). Hence, for any tensor $S$ of the appropriate type, each tensor $h'_{ab} = h_{ab;c\ldots d}S^{c\ldots d}$ (for the appropriate number of derivatives) satisfies (9) for each such $F$ and hence the appropriate equation in the set (10*a*) to (10*d*). From the arbitrariness of $S$ one obtains the desired expressions for these covariant derivatives. For example, if the curvature class at $m$ is $C$, then, at $m$, and if (7.1) and (7.2) are assumed

$$h_{ab;c} = h_{ab}\alpha_c + k_a k_b \beta_c, \qquad h_{ab;cd} = h_{ab}\alpha_{cd} + k_a k_b \beta_{cd} \tag{11}$$

for covectors $\alpha_c$ and $\beta_c$ and tensors $\alpha_{cd}$ and $\beta_{cd}$ at $m$.

It is also remarked that the classification made in section 2 is in fact independent of the *fitting metric* $h$ at $p \in M$. This is clear, by definition, for classes $O$, $C$ and $D$ and follows from (10*b*) for class $B$ and hence for class $A$. The classification could be done in a way that is manifestly independent of $h$ (but not of the existence of such an $h$), but the one given is more transparent. The sub-classification of $B$ into $B_1 \cup B_2$ does, however, depend on $h$, but this does not affect the results that follow.

These ideas can now be applied to the open subsets $A$, $\operatorname{int} B_1$, $\operatorname{int} B_2$, $\operatorname{int} C$ and $\operatorname{int} D$ in the decomposition (6).

**Theorem 2.** *Let $M$ be a smooth four-dimensional connected manifold admitting a smooth symmetric linear connection $\Gamma$ and associated curvature $R$ and also admitting a smooth Lorentz metric $h$ such that (2) holds. Then*

*(i) if, on $A$, (3.1) also holds, $\Gamma$ is compatible with a local Lorentz metric on $A$,*
*(ii) if, on* $\operatorname{int} B_1$ *(respectively* $\operatorname{int} B_2$*), (3.1) also holds, $\Gamma$ is compatible with a local Lorentz metric on* $\operatorname{int} B_1$ *(respectively* $\operatorname{int} B_2$*),*
*(iii) if, on* $\operatorname{int} C$*, (3.1) and (3.2) also hold, $\Gamma$ is compatible with a local Lorentz metric on an open dense subset of* $\operatorname{int} C$*,*
*(iv) if, on* $\operatorname{int} D$*, (3.1), (3.2) and (3.3) also hold, $\Gamma$ is compatible with a local Lorentz metric on an open dense subset of* $\operatorname{int} D$*.*

**Proof.** (i) The remarks preceding the theorem (see (10*a*)) together with the imposition of (3.1) or, equivalently, (7.1) show that $h_{ab;c} = h_{ab}w_c$ for some 1-form $w$ on the open subset $A$ which is easily seen to be smooth since $h$ and $\Gamma$ are. Now use (2) and the Ricci identity on $h$ to get

$$h_{ab;[cd]} = 0 \qquad (\Rightarrow h_{ab}w_{[c;d]} = 0). \tag{12}$$

Thus $w_{[c;d]} = 0$ and so $w_a$ is locally a gradient. Hence, for each $m \in A$, there is an open neighbourhood $W(\subseteq A)$ of $m$ on which $w_a = w_{,a}$ for some smooth function $w : W \to \mathbf{R}$. Then on $W$, $g_{ab} = \mathrm{e}^{-w}h_{ab}$ satisfies $g_{ab;c} = 0$ and is a local Lorentz metric for $\Gamma$. Further, if $g'$ is any other local metric defined on some neighbourhood $W'$ of $m$ and compatible with $\Gamma$, then $g'$ satisfies (2) on $W'$ and hence, on $W \cap W'$, $g' = \phi g$ for some positive smooth function $\phi : W \cap W' \to \mathbf{R}$ (see (10*a*)). From this and the result $g'_{ab;c} = 0$ it follows that $g'$ is a constant multiple of $g$ on $W \cap W'$.

(ii) The remarks preceding the theorem (see (10*b*)) together with the imposition of (3.1) and (7.1) show that for each $m \in \operatorname{int} B_1$ (respectively, $m \in \operatorname{int} B_2$) there is a coordinate neighbourhood $U \subseteq \operatorname{int} B_1$ (respectively, $U \subseteq \operatorname{int} B_2$) such that on $U$

$$h_{ab;c} = h_{ab}w_c + 2l_{(a}n_{b)}\lambda_c, \tag{13}$$

where $l$ and $n$ are the smooth null members of the null tetrad $l, n, x, y$ on $U$ established in section 2 and $w$ and $\lambda$ are (necessarily) smooth 1-forms on $U$. Since the contravariant components of $h$ satisfy $h^{ac}h_{cb} = \delta^a{}_b$, the result $(h^{ac}h_{cb})_{;d} = 0$ implies that

$$h^{ab}{}_{;c} = -h^{ab}w_c - 2l^{(a}n^{b)}\lambda_c. \tag{14}$$

Now $U$ can be chosen so that, in addition to the above results, the curvature tensor satisfies (5) on $U$ with $\alpha, \beta$ smooth nowhere zero functions and $F$ and $G$ as given in (5) in terms of the null tetrad on $U$. The Bianchi identities between $R$ and $\Gamma$ are $R^a{}_{b[cd;e]} = 0$. One now employs a technique first used in [16] (where *h was*, in fact, a metric compatible with $\Gamma$). First, one writes the Bianchi identity using (5) and contracts with $l_a x^b l^c n^d x^e$. This operation eliminates all terms except one and gives, on $U$

$$\beta G^a{}_{b;e} G_{cd} l_a x^b l^c n^d x^e = 0 \qquad (\Rightarrow l_{a;b} x^a x^b = 0). \tag{15}$$

Similar contractions with $l_a x^b l^c n^d y^e, l_a x^b x^c y^d l^e$ and $l_a x^b x^c y^d n^e$ (and the result $l^a y_{a;b} = -y^a l_{a;b}$, which follows from applying (14) to the expansion of $(h^{ab} l_a y_b)_{;c} = 0$) lead to

$$l_{a;b} x^a y^b = l_{a;b} y^a l^b = l_{a;b} y^a n^b = 0. \tag{16}$$

Observations of the symmetry in these contractions then give

$$l_{a;b} y^a y^b = l_{a;b} y^a x^b = l_{a;b} x^a l^b = l_{a;b} x^a n^b = 0. \tag{17}$$

From these results it easily follows that $l_{a;b} x^a = l_{a;b} y^a = 0$. Further, one has $(h_{ab} l^a l^b)_{;c} = 0$ and so, using (13), one has $l^a_{;b} l_a = 0$ and hence $l_{a;b} l^a = 0$. It follows that $l_{a;b} = l_a p_b$ for some smooth 1-form $p$ on $U$ and then, from (14), that $l^a{}_{;b} = l^a (p_b - w_b - \lambda_b)$. Thus $l^a$ and $l_a$ are *recurrent* on $U$. Similar arguments show that $n^a$ and $n_a$ are *recurrent* on $U$ and, since $(l^a n_a)_{;b} = 0$, that $n_{a;b} = n_a (\lambda_b + w_b - p_b)$ and $n^a{}_{;b} = -n^a p_b$. It is then easily checked that the nowhere zero symmetric tensor $T_{ab} = 2 l_{(a} n_{b)}$ on $U$ is also recurrent in the sense that $T_{ab;c} = T_{ab} r_c$, where $r$ is the 1-form $r_a = \lambda_a + w_a$ on $W$. When the Ricci identity is applied to $T_{ab}$ and use is made of (5) one gets

$$T_{ab;[cd]} = 2 n_e l_{(a} R^e{}_{b)cd} + 2 l_e n_{(a} R^e_{b)cd} (= T_{ab} r_{[c;d]}) = 0 \tag{18}$$

and so $r_{[a;b]} = 0$. Thus one may assume, by shrinking $W$ if necessary, that, on $W$, $r_a = r_{,c}$ for a smooth function $r : W \to \mathbf{R}$. It follows that, on $W$, the tensor $t_{ab} = \mathrm{e}^{-r} T_{ab}$ is symmetric, nowhere zero and covariantly constant, $t_{ab;c} = 0$. When (13) is rewritten as $h_{ab;c} = h_{ab} w_c + t_{ab} \gamma_c$ for some smooth 1-form $\gamma_a$ on $W$ and the Ricci identity is applied to $h$ (using (2)) one finds

$$h_{ab} w_{[c;d]} + t_{ab} (\gamma_{[c;d]} - \gamma_{[c} w_{d]}) = 0. \tag{19}$$

At each point of $W$ the matrices $h_{ab}$ and $t_{ab}$ have (different) ranks 4 and 2, respectively, and so the two quantities on the left-hand side of (19) must vanish separately. The first of these equations leads, as before, to the fact that $w_a$ is locally a gradient and the second gives $\gamma_{[c;d]} - \gamma_{[c} w_{d]} = 0$. Thus by again shrinking $W$ if necessary one has $w_a = w_{,a}$ on $W$ and then the second condition can be rewritten as

$$(\mathrm{e}^{-w} \gamma_a)_{;b} - (\mathrm{e}^{-w} \gamma_b)_{;a} = 0. \tag{20}$$

This means, by again reducing $W$ if necessary, that there exists a smooth function $\delta : W \to \mathbf{R}$ such that $\gamma_a = \mathrm{e}^w \delta_{,a}$. Now consider a tensor $g$ on $W$ of the form

$$g_{ab} = \phi h_{ab} + \epsilon t_{ab} \tag{21}$$

for smooth functions $\phi, \epsilon : W \to \mathbf{R}$ with $\phi$ nowhere zero. Then, by again considering rank on the indices $a, b$, one sees that the condition that $g$ is covariantly constant, $g_{ab;c} = 0$, is equivalent to the two differential equations

$$\phi_{,a} + \phi w_{,a} = 0 \qquad \epsilon_{,a} + \phi\, \mathrm{e}^w \delta_{,a} = 0, \tag{22}$$

which are to be regarded as equations to find $\phi$ and $\epsilon$ with $w$ and $\delta$ given. The general solutions are $\phi = D\,\mathrm{e}^{-w}$ and $\epsilon = C - D\delta$ for $C, D \in \mathbf{R}$ and so if $g_{ab} = D\mathrm{e}^{-w} h_{ab} + (C - D\delta) t_{ab}$, then $g_{ab;c} = 0$. Further, it is clear that $C$ and $D$ may be chosen so that $g$ is non-degenerate at $m$ and hence on a (possibly reduced) open neighbourhood $W$ of $m$. It is also clear that the original null tetrad $l, n, x, y$ (with respect to $h$) is, after a possible smooth rescaling of these tetrad members, a null tetrad with respect to $g$ on $W$ and so $g$ is a Lorentz metric on $W$ compatible with $\Gamma$. This construction of $g$ and (10$b$) shows that any Lorentz metric compatible with $\Gamma$ is of this general form on some open neighbourhood of $m$.

(iii) The assumption that (2), (3.1) and (3.2) hold on $\operatorname{int} C$ means that (7.1) and (7.2) also hold on $\operatorname{int} C$ and hence that for $m \in \operatorname{int} C$, there is an open neighbourhood $U$ of $m$ and a smooth nowhere zero vector field $k$ on $U$ such that

$$h_{ab;c} = h_{ab} w_c + k_a k_b \lambda_c \tag{23}$$

$$h_{ab;cd} = h_{ab} X_{cd} + k_a k_b Y_{cd} \tag{24}$$

for necessarily smooth 1-forms $w$ and $\lambda$ and tensors $X$ and $Y$. If (23) is covariantly differentiated and equated with (24) and the resulting equation contracted at each $m \in U$ with $t^a t^b$, where $t \in T_m M$ is such that $h_{ab} t^a t^b \neq 0$, $h_{ab} k^a t^b = 0$, one finds that on $U$

$$X_{ab} = w_a w_b + w_{a;b}. \tag{25}$$

When this information is replaced in the equation from which it came, one finds, after cancellation, that

$$k_a (k_b Y_{cd} - k_b \lambda_{c;d} - k_b w_c \lambda_d - k_{b;d} \lambda_c) = k_{a;d} k_b \lambda_c. \tag{26}$$

The value of $\lambda$ in (23) depends, of course, on the choice of $k$ and $U$ there being the freedom to replace $k$ by $\mu k$ for some nowhere zero smooth function $\mu : U \to \mathbf{R}$. However, the condition that $\lambda(m) = 0$ is independent of the choice of $k$ and $U$ and characterizes the point $m \in \operatorname{int} C$. So let $C_1$ (respectively $C_2$) be the subset of points of $\operatorname{int} C$ at which this condition holds (respectively, does not hold). Then $C_2$ is open in $\operatorname{int} C$ and hence in $M$ and one has a disjoint decomposition of $\operatorname{int} C$ in the form

$$\operatorname{int} C = C_1 \cup C_2 = \operatorname{int} C_1 \cup C_2 \cup Z_1, \tag{27}$$

where $\operatorname{int} C_1$ is the interior of $C_1$ in either the topology of $M$ or the subspace topology of $\operatorname{int} C$ (these being equal). The subset $Z_1$ is defined by the disjointness of the decomposition (27) and hence, since $Z_1 \subseteq C_1$, $Z_1$ has empty interior in either $M$ or $\operatorname{int} C$. Thus $\operatorname{int} C_1 \cup C_2$ is open in $M$ and open and dense in $\operatorname{int} C$. It follows from (26) that $k$ is *recurrent* on $C_2$, $k_{a;b} = k_a p_b$, where $p$ is now a smooth 1-form on $C_2$.

On $\operatorname{int} C_1$, $\lambda = 0$ and so (23) gives $h_{ab;c} = h_{ab} w_c$. A similar argument to one earlier then shows that each $m \in \operatorname{int} C_1$ admits an open neighbourhood $W$ on which $w_a = w_{,a}$ and on which $g_{ab} = \mathrm{e}^{-w} h_{ab}$ is then a Lorentz metric compatible with $\Gamma$.

On $C_2$, the recurrence of $k$ together with the properties of class $C$ and the Ricci identity give

$$k_{a;[bc]} = 0 \qquad (\Rightarrow k_a p_{[b;c]} = 0 \Rightarrow p_{[a;b]} = 0) \tag{28}$$

and so $p_a$ is locally a gradient. Thus each $m \in C_2$ admits a neighbourhood $W$ on which $p_a = p_{,a}$ for some smooth function $p : W \to \mathbf{R}$. Thus on $W$ the 1-form $k'_a = \mathrm{e}^{-p} k_a$ is covariantly constant, $k'_{a;b} = 0$. Now return to (23), which can be assumed to hold on $W$ with $k_a$ replaced by $k'_a$ (and the prime on $k'$ dropped), construct $h_{ab;cd}$ and skew symmetrize over the indices $c$ and $d$ (using $h_{ab;[cd]} = 0$ from (2)) to get

$$h_{ab} w_{[c;d]} + k_a k_b (\lambda_{[c;d]} - \lambda_{[c} w_{d]}) = 0. \tag{29}$$

Now one proceeds in a similar manner to that in part (ii) to get the relations

$$w_{[a;b]} = 0 \qquad \lambda_{[a;b]} - \lambda_{[a} w_{b]} = 0. \tag{30}$$

From this one has on $W$ (by reducing $W$ if necessary) a function $w : W \to \mathbf{R}$ such that $w_a = w_{,a}$ and a relation like (20) with $\gamma_a$ replaced by $\lambda_a$ and hence a function $\delta : W \to \mathbf{R}$ such that $\lambda_a = \mathrm{e}^w \delta_{,a}$. The potential metrics on $W$ are thus of the form

$$g_{ab} = \phi h_{ab} + \epsilon k_a k_b \tag{31}$$

for necessarily smooth functions $\phi, \epsilon : W \to \mathbf{R}$ and with $\phi$ positive which, from $g_{ab;c} = 0$, are easily seen to satisfy the same conditions as $\phi$ and $\epsilon$ do in (22). Thus the general solution is $\phi = D\,\mathrm{e}^{-w}$ and $\epsilon = C - D\delta$ for $C, D \in \mathbf{R}$ chosen to preserve Lorentz signature on a (possibly reduced) open subset of $W$ (and the signature is automatically Lorentz at those points of $W$, where $k$ is null with respect to $h$). Thus $g$ is a Lorentz metric on $W$ compatible with $\Gamma$.

(iv) For this part one assumes that (2) together with (3.1), (3.2) and (3.3) holds. This is equivalent to assuming (2) together with (7.1), (7.2) and (7.3). Thus if $m \in \operatorname{int} D$ there are smooth covector fields $r$ and $s$ defined on some open neighbourhood $U$ of $m$ such that $r(m')$ and $s(m')$ are independent members of $T^*_{m'}M$ at each $m' \in U$ (section 2) and such that, on $U$,

$$h_{ab;c} = h_{ab}w_c + r_a r_b \beta_c + s_a s_b \gamma_c + 2r_{(a}s_{b)}\delta_c \tag{32}$$

$$h_{ab;cd} = h_{ab}X_{cd} + r_a r_b Y_{cd} + s_a s_b Z_{cd} + 2r_{(a}s_{b)}W_{cd} \tag{33}$$

$$h_{ab;cde} = h_{ab}X_{cde} + r_a r_b Y_{cde} + s_a s_b Z_{cde} + 2r_{(a}s_{b)}W_{cde}, \tag{34}$$

where $w_a, \beta_a, \ldots, W_{abc}$ are smooth tensors of the appropriate type on $U$. Now, by shrinking $U$ if necessary, one may assume that, in addition to the smooth covector fields $r$ and $s$ on $U$, there are smooth covector fields $e$ and $f$ on $U$ such that $r(m'), s(m'), e(m')$ and $f(m')$ are independent members of $T^*_{m'}M$ at each $m' \in U$. Then one defines a *positive definite* metric $\gamma$ on $U$ by $\gamma_{ab} = r_a r_b + s_a s_b + e_a e_b + f_a f_b$ (and where $\gamma^{ab}$ temporarily denotes the inverse of $\gamma_{ab}$ and *not* indices raised with metric $h$). Then define smooth vector fields $R, S, E$ and $F$ on $U$ by $R^a = \gamma^{ab}r_b, \ldots, F^a = \gamma^{ab}f_b$ and smooth covector fields $p, q, p'$ and $q'$ on $U$ by $p_a = R^b r_{b;a}, q_a = S^b r_{b;a}, p'_a = R^b s_{b;a}$ and $q'_a = S^b s_{b;a}$. Finally, define two type $(0, 2)$ tensor fields $u$ and $u'$ on $U$ by

$$r_{a;b} = r_a p_b + s_a q_b + u_{ab} \tag{35}$$

$$s_{a;b} = r_a p'_b + s_a q'_b + u'_{ab}. \tag{36}$$

Thus $R, S, E$ and $F$ form a $\gamma$-orthogonal tetrad at each point of $U$ and a contraction of each of (35) and (36) with $R^a$ and with $S^a$ gives

$$R^a u_{ab} = R^a u'_{ab} = S^a u_{ab} = S^a u'_{ab} = 0. \tag{37}$$

Now take the covariant derivative of (32) using (35) and (36) and equate to (33). (It is noted at this point that the above construction of the vector fields $R, S, E$ and $F$ may be achieved whilst, at the same time, ensuring that $E$ is nowhere null on $U$ *with respect to* $h$ (that is, $h_{ab}E^aE^b$ is nowhere zero on $U$).) A contraction of the resulting equation with $E^aE^b$ gives

$$X_{ab} = w_{a;b} + w_a w_b \tag{38}$$

whilst successive contractions of this same equation with $R^aR^b$, $S^aS^b$ and $R^aS^b$, using (37) and taking into account (38), give

$$\begin{aligned} Y_{ab} &= w_a\beta_b + 2\beta_a p_b + 2\delta_a p'_b + \beta_{a;b} \\ Z_{ab} &= w_a\gamma_b + 2\gamma_a q'_b + 2\delta_a q_b + \gamma_{a;b} \\ W_{ab} &= w_a\delta_b + \beta_a q_b + \gamma_a p'_b + \delta_a q'_b + \delta_a p_b + \delta_{a;b}. \end{aligned} \tag{39}$$

A back substitution of (39) into the equation from which they arose and use of (35) and (36) then give after a long but straightforward calculation

$$r_{(a}u_{b)d}\beta_c + s_{(a}u_{b)d}\delta_c + r_{(a}u'_{b)d}\delta_c + s_{(a}u'_{b)d}\gamma_c = 0. \tag{40}$$

Finally, contractions of (40) with $R^a$ and $S^a$ using (37) give, respectively, at $m$

$$u_{bd}\beta_c + u'_{bd}\delta_c = 0 \qquad u'_{bd}\gamma_c + u_{bd}\delta_c = 0. \tag{41}$$

Now given $m \in U$, as above, the covector fields $r$ and $s$ are determined up to changes $r \to r'$ and $s \to s'$ where

$$r' = \rho r + \sigma s \qquad s' = \rho' r + \sigma' s, \tag{42}$$

where $\rho, \sigma, \rho'$ and $\sigma'$ are smooth functions: $U \to \mathbf{R}$ (with $U$ possibly reduced) such that $r'(m')$ and $s'(m')$ are independent at each $m' \in U$. The covector field $w$ in (32) is clearly independent of the choice of $r$ and $s$ but the covector fields $\beta$, $\gamma$ and $\delta$ in (32) are not. However, the condition that $\beta(m) = \gamma(m) = \delta(m) = 0$ is independent of the choice of $r$ and $s$ and $U$ and (cf case (iii)) is characteristic of $m \in \operatorname{int} D$. Let $D_1$ (respectively, $D_2$) be the subset of $\operatorname{int} D$ of those points where this condition is satisfied (respectively, not satisfied). Then $\operatorname{int} D_1$ and $D_2$ are open in $\operatorname{int} D$ and hence in $M$. Thus $\operatorname{int} D$ admits the disjoint decomposition $\operatorname{int} D = \operatorname{int} D_1 \cup D_2 \cup Z_2$, where $Z_2$ has empty interior in $\operatorname{int} D$ and in $M$. Next, let $m \in D_2$. The tensors $u$ and $u'$ at $m$ depend on the choice of $r$ and $s$ from (35), (36) and (42). However, it is straightforward to check that the condition that $u(m) = u'(m) = 0$ is independent of the choice of $r$ and $s$ and $U$ and is again characteristic of $m$. So let $D_3$ (respectively, $D_4$) be the subset of $D_2$ of those points where this condition is satisfied (respectively, not satisfied). Then $\operatorname{int} D_3$ and $D_4$ are open subsets of $D_2$ and hence of $M$ and one has a disjoint decomposition of $D_2$ given by $D_2 = \operatorname{int} D_3 \cup D_4 \cup Z_3$, where $Z_3$ has empty interior in $D_2$ and hence in $M$. Thus one has a disjoint decomposition of $\operatorname{int} D$ given by

$$\operatorname{int} D = \operatorname{int} D_1 \cup \operatorname{int} D_3 \cup D_4 \cup Z_4, \tag{43}$$

where $Z_4 = Z_2 \cup Z_3$. Now $Z_3 \subseteq D_2$ and $Z_2 \cap D_2 = \emptyset$ and so if $V (\neq \emptyset)$ is open in $M$ with $V \subseteq Z_4$, then, since $Z_2$ and $Z_3$ have empty interiors in $M$, $V$ cannot be contained in either $Z_2$ or $Z_3$ and hence $V \cap Z_2 \neq \emptyset \neq V \cap Z_3$. But since $D_2$ is open, $V \cap D_2$ is open and a subset of $Z_3$ and is hence empty. Now $V \cap Z_3 \subseteq V \cap D_2$ and so one achieves the contradiction that $V \cap Z_3 = \emptyset$. It follows that $V = \emptyset$ and so $Z_4$ has empty interior in $M$.

Let $m \in \operatorname{int} D_1$ so that, from (32), $h_{ab;c} = h_{ab}w_c$ on some open neighbourhood $U$ of $m$. Then, by reducing $U$ if necessary, one has, as before (see (12)), a smooth function $w : U \to \mathbf{R}$ such that $w_a = w_{,a}$ and then $\mathrm{e}^{-w}h_{ab}$ is a Lorentz metric on $U$ compatible with $\Gamma$.

Let $m \in \operatorname{int} D_3$ so that, whichever covector fields $r$ and $s$ are chosen, (35) and (36) hold on some open neighbourhood $U$ of $m$ with $u$ and $u'$ zero on $U$. Then it follows from these equations and the definition of the curvature class $D$ that

$$\begin{aligned} &r_a R^a_{bcd} = 0, \qquad && r_a R^a_{bcd;f_1\ldots f_n} = 0 \quad (n = 1, 2, \ldots) \\ &s_a R^a{}_{bcd} = 0, \qquad && s_a R^a{}_{bcd;f_1\ldots f_n} = 0 \quad (n = 1, 2, \ldots). \end{aligned} \tag{44}$$

Thus the infinitesimal holonomy group of $\Gamma$ is one-dimensional at each $m' \in U$ and, given that $U$ is chosen connected and simply connected (as it always can be), is isomorphic (as a Lie group) to the holonomy group of $\Gamma$ (restricted to $U$) and with holonomy algebra represented by the matrices in the range of the map $f$ described in section 2. Since (2) holds this holonomy group is a subgroup of the orthogonal group of $h$ and it follows that a metric on $U$ compatible with $\Gamma$ is now assured [2] (see, also [1]). To actually construct such a metric one notes that the members of $T_m M$ associated with $r(m)$ and $s(m)$ under $h$ annihilate each matrix in the above representation of the holonomy algebra. Hence, by exponentiation, they give rise to holonomy invariant (i.e. covariantly constant) vector fields on $U$ which span the same distribution on $U$ as do the vector fields associated with $r$ and $s$ under $h$ (see, e.g., [1, 4]). Hence one may now suppose, using the freedom (42), that $r$ and $s$ above are covariantly constant. In this case, covariantly differentiating (32) and using the Ricci identity for $h$ and (2), one finds

$$w_{[a;b]} = 0, \qquad \beta_{[a;b]} = \beta_{[a} w_{b]}, \qquad \gamma_{[a;b]} = \gamma_{[a} w_{b]}, \qquad \delta_{[a;b]} = \delta_{[a} w_{b]} \tag{45}$$

and so since $U$ is simply connected, $w_a = w_{,a}$ with $w : U \to \mathbf{R}$. Then (45) shows that $e^{-w}\beta$, $e^{-w}\gamma$ and $e^{-w}\delta$ are closed 1-forms on $U$ and so there exist $\mu, \nu, \lambda : U \to \mathbf{R}$ such that $\beta_a = e^{w}\mu_{,a}$, $\gamma_a = e^{w}\nu_{,a}$ and $\delta_a = e^{w}\lambda_{,a}$. Now construct the tensor $g$ on $U$ by

$$g_{ab} = e^{-w} h_{ab} + (c_1 - \mu) r_a r_b + (c_2 - \nu) s_a s_b + 2(c_3 - \lambda) r_{(a} s_{b)}, \tag{46}$$

where $c_1, c_2, c_3 \in \mathbf{R}$. It is easily checked that $g$ is covariantly constant and, by appropriate choice of $c_1$, $c_2$ and $c_3$ (e.g. $c_1(m) = \mu(m)$, $c_2(m) = \nu(m)$ and $c_3(m) = \lambda(m)$), $g$ can be made to have Lorentz signature in some neighbourhood of $m$.

Now let $m \in D_4$ so that at least one of $\beta(m)$, $\gamma(m)$ and $\delta(m)$ is not zero. Then, by using the freedom permitted by (42), one may arrange that $\beta$, $\gamma$ and $\delta$ are each non-zero at $m$ and hence in some open neighbourhood $U \subseteq D_4$ of $m$. It then follows from (41) and the definition of the region $D_4$ that $u$ and $u'$ are nowhere zero and proportional on $U$ so that, on $U$, $u' = \kappa u$ with $\kappa : U \to \mathbf{R}$ nowhere zero. Now one can again use the freedom in (42) by replacing $r$ by $r' = \kappa r - s$ (with $s$ unchanged) to achieve (35) and (36) with $u$ zero and $u'$ nowhere zero on $U$. It then follows from (41) that $\gamma$ and $\delta$ vanish on $U$ (and hence that $\beta$ is nowhere zero on $U$). Then (32) is

$$h_{ab;c} = h_{ab} w_c + r_a r_b \beta_c, \tag{47}$$

where, for convenience, the prime on $r$ is omitted. From (47) and (35) (with $u = 0$) one then finds

$$h_{ab;cd} = h_{ab}(w_{c;d} + w_c w_d) + r_a r_b(\beta_{c;d} + w_c \beta_d + 2\beta_c p_d) + 2r_{(a} s_{b)} \beta_c q_d. \tag{48}$$

At this point the extra condition (3.3) (or, equivalently (7.3) or (34)) is introduced. Thus one takes one more covariant derivative of (48) and equates it to the right-hand side of (34) to obtain an equation on $U$ of the form

$$h_{ab} A_{cde} + r_a r_b B_{cde} + s_a s_b C_{cde} + 2r_{(a} s_{b)} D_{bde} + 2r_{(a} u'_{b)e} \beta_c q_d = 0, \tag{49}$$

where the tensors $A, B, C$ and $D$ can be calculated but whose exact form is not needed. A contraction with $E^a E^b$ (cf the proof of (38)) gives $A = 0$ on $U$ whilst contractions with $R^a R^b$, $S^a S^b$ and $R^a S^b$ reveal that $B = C = D = 0$ on $U$. Thus, on $U$, (49) becomes

$$r_a u'_{be} \beta_c q_d + r_b u'_{ae} \beta_c q_d = 0. \tag{50}$$

A contraction of (50) with $R^a$ and the use of (37) then reveals that $q_a = 0$ on $U$ (since $u'$ and $\beta$ are nowhere zero on $U$). Thus from (35), $r_{a;b} = r_a p_b$ and then the Ricci identity on $r$ reveals that $p_{[a;b]} = 0$ and so, by shrinking $U$ if necessary, one has $p_a = p_{,a}$ for some function $p : U \to \mathbf{R}$. Then one can replace $r$ in the above by $e^{-p} r$, where $(e^{-p} r_a)_{;b} = 0$. This latter (covariantly constant) vector field will still be labelled $r$. Then (47) and (2) show that $w_{[a;b]} = 0$ and (similar to case (iii)) $\beta_{[a;b]} - \beta_{[a} w_{b]} = 0$. Thus, again by reducing $U$ if necessary one has functions $w, \theta : U \to \mathbf{R}$ such that $w_a = w_{,a}$ and $\beta_a = e^{w}\theta_{,a}$. Then tensor $g$ defined on $U$ by

$$g_{ab} = e^{-w} h_{ab} - (\theta + C) r_a r_b \tag{51}$$

is now easily checked to be covariantly constant, where the arbitrary constant $C$ can be adjusted to ensure that $g$ has Lorentz signature in some open neighbourhood of $m$.

In summary, the manifold $M$ has been disjointly decomposed as $M = M' \cup Z'$, where $M'$ is the open subset of $M$ given by

$$M' = A \cup \operatorname{int} B_1 \cup \operatorname{int} B_2 \cup \operatorname{int} C_1 \cup C_2 \cup \operatorname{int} D_1 \cup \operatorname{int} D_3 \cup D_4 \cup \operatorname{int} O \tag{52}$$

each point of which (under the appropriate conditions in theorem 2) admits an open neighbourhood and a metric on that neighbourhood compatible with $\Gamma$ (this being clearly true with no further conditions on the region int $O$) and where $Z' = Z \cup Z_1 \cup Z_4$. Now each of $Z$, $Z_1$ and $Z_4$ has empty interior in $M$ and $Z$ is closed in $M$. Also $Z_1 \subseteq \operatorname{int} C$ and $Z_4 \subseteq \operatorname{int} D$ and so (since, from (6), $Z$ is disjoint from int $C$ and int $D$) $Z$, $Z_1$ and $Z_4$ are mutually disjoint. It now follows that $Z'$ is a closed subset of $M$ with empty interior and hence that $M'$ is open and dense in $M$. To see this one first notes that $Z'$ is closed by definition and that from the preliminary decompositions (6) and (27), $Z \cup Z_1$ is closed. Now let $\emptyset \neq V \subseteq Z'$ with $V$ open. Then $V' \equiv V \cap (M \backslash Z)$ is open and $V' \cap Z = \emptyset$. Then $V'' \equiv V' \cap (M \backslash (Z \cup Z_1))$ is open and $V'' \subseteq Z_4$. Thus $V'' = \emptyset$ (since $\operatorname{int} Z_4 = \emptyset$) and hence $V' \subseteq Z \cup Z_1$. Thus $V' \subseteq Z_1$ and so $V' = \emptyset$. This means that $V \subseteq Z$ which is a contradiction (since $\operatorname{int} Z = \emptyset$). Thus $\operatorname{int} Z' = \emptyset$ and $M'$ is open and dense in $M$. This completes the proof of theorem 2. □

The next two theorems are consequences of theorem 2.

**Theorem 3.** *Under the conditions of theorem 2, if $\Gamma$ is Ricci flat, then $D = \emptyset$ and equations (2), (3.1) and (3.2) on $M$ are sufficient to ensure a local Lorentz metric compatible with $\Gamma$ on the open subset $M'$ of $M$.*

**Proof.** If the curvature class at $m \in M$ is $D$, then the curvature tensor takes the form $R_{abcd}(= h_{ae} R^e_{bcd}) = \lambda F_{ab} F_{cd}$ for some simple bivector $F$ at $m$ and $0 \neq \lambda \in \mathbf{R}$ (section 2). The Ricci flat condition $R^c{}_{bcd} = 0$ then gives $F^c{}_b F_{cd} = 0$. Since $F$ is simple one may write $F = p \wedge q$ for $p, q \in T^*_m M$ and with $p$ and $q$ ($h$-)orthogonal. The Ricci flat condition then shows that $p$ and $q$ are each ($h$-)null, contradicting the Lorentz signature of $h$. Thus $D = \emptyset$ and the result now follows from theorem 2. □

**Theorem 4.**

*(i) If $m \in M'$, with $M'$ given by (52) and if $h$ satisfies (2) and those equations in (3) appropriate to the region of $m'$ in which $m'$ is located, and as given in theorem 2, in that region, then $h$ satisfies (3.k) for all $k$ at each point of that region.*

*(ii) If the infinitesimal holonomy algebra associated with $\Gamma$ has constant dimension on $M$ and $M$ is simply connected and if the conditions appropriate for each of the regions as specified in theorem 2 hold in these regions, $M$ admits a global Lorentz metric which is compatible with $\Gamma$.*

**Proof.** (i) The proof of this follows rather quickly by using the various relationships obtained in the proof of theorem 2 between $h$ and the constructed local metric $g$. Thus $h$ and $g$ were either conformally related or linked by various recurrent tensors $t$ or recurrent 1-forms $k$. The result now follows from (1).

(ii) The constancy of the dimension of the infinitesimal holonomy algebra means that the infinitesimal holonomy groups at each $m \in M$ are isomorphic to each other and to the restricted holonomy group of $M$ and hence (since $M$ is simply connected) to the holonomy group of $M$ [1]. Then part (i) shows that $h$ satisfies (3.$k$), for all $k$, on $M'$ and hence on $M$. Thus the infinitesimal holonomy algebra at each $m \in M$ is a subalgebra of the orthogonal algebra of $h$ (since each contribution to the infinitesimal holonomy algebra satisfies (4)). It follows that the holonomy group of $M$ is isomorphic to a subgroup of the orthogonal group of $h$

(that is, its members preserve the quadratic form $h$ at each $m \in M$ [2]—see, also, [1] and section 1). The result in (ii) now follows. (In fact the original conditions (2) and (3.1)–(3.$k$) imposed may be regarded as forcing a certain subspace of the infinitesimal holonomy algebra to lie within the orthogonal algebra of $h$.) □

## 4. Remarks and examples

In this paper the tensor $h$ was assumed to be of Lorentz signature. However, it is clear that if $h$ is assumed positive-definite, the curvature types $A$, $B$, $C$, $D$ and $O$ still make sense after an obvious change for type $B$ where the range of $f$ would be two-dimensional and spanned, in an obvious notation, by $x^a y_b - y^a x_b$ and $z^a w_b - w^a z_b$, where $x$, $y$, $z$, $w$ is an orthonormal tetrad at the point in question. Also, even when $h$ is taken as having Lorentz signature, it has been pointed out that metrics of both Lorentz *and* positive-definite signature can sometimes be constructed which are compatible with $\Gamma$ (cf [4]).

The results of theorem 2 give sufficient conditions in terms of equations (2) and (3.1)–(3.$k$) for each curvature class for a connection to be metric. The work of Edgar shows that for a fairly general class of cases condition (2) is sufficient, but the class to which this applies is somewhat obscure [8]. Here some examples will be given which show that (2) is not a sufficient condition for the existence of a local metric for any of the curvature classes $A$, $B$, $C$ or $D$.

**Example 1** (This example is taken from [17])**.** Let $(\mathrm{R}^4, \eta)$ denote Minkowski space and let $U$ be the open submanifold $t > 0$ of $\mathrm{R}^4$. Define $\phi : U \to \mathrm{R}$ by $\phi(x, y, z, t) = \log t$ so that $\phi_{a;b} = -\phi_a\phi_b$, where $\phi_a = \phi_{,a}$. Define a metric $g$ on $U$ by $g = \mathrm{e}^{-2\phi}\eta$ and let $\Gamma$ and $R$ be the associated Levi-Civita connection and curvature tensor, respectively. Then in the coordinates on $U$ inherited from $\mathrm{R}^4$

$$\Gamma^a{}_{bc} = -(\delta^a{}_b\phi_c + \delta^a{}_c\phi_b - \phi^a g_{bc}) \tag{53}$$

$$R^a{}_{bcd} = \phi^e\phi_e(\delta^a{}_d g_{bc} - \delta^a{}_c g_{bd}), \tag{54}$$

where $\phi^a = g^{ab}\phi_b$. Then define a 1-form $\psi$ on $U$ by $\psi_a = (1-\mathrm{e}^{\phi})^{-1}\phi_a$ and a new symmetric connection $\Gamma'$ on $U$ by the components (cf [18])

$$\Gamma'^a{}_{bc} = \Gamma^a{}_{bc} + \delta^a{}_b\psi_c + \delta^a{}_c\psi_b. \tag{55}$$

This has the curvature tensor $R'$ with components

$$R'^a{}_{bcd} = \phi^e\phi_e(1 - (1-\mathrm{e}^{\phi})^{-1})(\delta^a{}_d g_{bc} - \delta^a{}_c g_{bd}). \tag{56}$$

It is easy to check that $R'$ and $g$ satisfy (2) and that the map $f$ associated with $R'$ (section 2) has rank equal to 6 everywhere on $U$ and so the curvature class is everywhere $A$. However, using $R'$ and $\Gamma'$ one can check after a short calculation that (3.1) fails. In fact $\Gamma'$ is not a local metric connection because if it were compatible with a metric $g'$ on some open subset $V$ of $U$, then from (10$a$) one would have without loss of generality $g' = \mathrm{e}^{\alpha}g$ for some smooth function $\alpha : V \to \mathrm{R}$. But $g$ has zero covariant derivative with respect to $\Gamma$ and so the condition that $g'$ has zero covariant derivative with respect to $\Gamma'$ gives, using (55),

$$\alpha_{,c}g_{ab} - 2g_{ab}\psi_c - g_{ac}\psi_b - g_{bc}\psi_a = 0. \tag{57}$$

On contracting (57) with $g^{ab}$ one finds $2\alpha_{,c} = 5\psi_c$ and a back substitution gives $g_{ab}\psi_c = 2g_{ac}\psi_b + 2g_{bc}\psi_a$. A final contraction at any $p \in V$ with $X^c$ for any $X \in T_pV$ and a consideration of rank reveals that $X^a\psi_a = 0$ for all such $X$ and hence the contradiction that

$\psi \equiv 0$ on $V$. Thus although $\Gamma'$ yields a curvature tensor of class $A$ satisfying (2), (3.1) fails and $\Gamma'$ is *not* locally metric.

**Example 2.** This example is quite general and can be applied to each of the curvature classes $A$, $B$, $C$ and $D$. Let $(M, g)$ be a spacetime which admits a nowhere zero *recurrent null* vector field $l$ with the property that no rescaling of $l$ is covariantly constant over some non-empty open subset of $M$. This means that at some $p \in M$, $R^a{}_{bcd}l^d \neq 0$, for otherwise, one has $l_{a;b} = l_a p_b$ for some covector field $p$ on $M$ which from the Ricci identity satisfies $p_{[a;b]} = 0$ on $M$. Thus for any $p \in M$ there is an open neighbourhood $U$ of $p$ and a function $\phi : U \to \mathrm{R}$ such that $p_a = \phi_{,a}$ and then $\mathrm{e}^{-\phi}l$ is a covariantly constant rescaling of $l$ over $U$. Thus there exists $p \in M$ and, by continuity, some open neighbourhood $V$ of $p$ on which $R^a{}_{bcd}l^d$ is nowhere zero. The recurrence condition and the Ricci identity on $l$ in fact show that $R^a{}_{bcd}l^d = F^a{}_b l_c$ for some tensor $F$ on $V$ which is hence nowhere zero on $V$. Restricting attention to $V$ with the original metric $g$ and associated Levi-Civita connection $\Gamma$ it is noted that $l_{[a}l_{b;c]} = 0$ and hence that $l$ is hypersurface orthogonal on $V$. Thus, shrinking $V$, if necessary, one may assume that $l$ is scaled to a gradient vector field $l'$ on $V$ such that $l'_{a;b} = \psi l'_a l'_b$ for some function $\psi : V \to R$. Now define a new symmetric connection $\Gamma'$ on $V$ by the components $\Gamma'^a_{bc} = \Gamma^a{}_{bc} + l'^a l'_b l'_c$. It is easily calculated that the curvature tensor $R'$ on $V$ arising from $\Gamma'$ is identical to the original one (this can be quickly checked by first noting that if $P^a{}_{bc} = \Gamma'^a{}_{bc} - \Gamma^a_{bc} = l'^a l'_b l'_c$ then $P^a{}_{bd;c} - P^a{}_{bc;d} = 0$ and $P^a_{bc}P^c{}_{de} = 0$ and then using a convenient formula in [19, 20]). Thus in any chart on $V$,

$$R'^a{}_{bcd} = R^a{}_{bcd}. \tag{58}$$

Using a vertical stroke to denote a $\Gamma'$ covariant derivative one thus has on $V$, using another convenient result in [19, 20],

$$\begin{aligned} R'^a{}_{bcd|e} &= R^a{}_{bcd|e} \\ &= R^a{}_{bcd;e} + R^f{}_{bcd}P^a{}_{fe} - R^a{}_{fcd}P^f{}_{be} - R^a{}_{bfd}P^f{}_{ce} - R^a{}_{bcf}P^f{}_{de} \\ &= R^a{}_{bcd;e} - 2l'^a l'_b l'_e F_{cd}. \end{aligned} \tag{59}$$

It now easily follows that for $R'^a{}_{bcd}$, $R'^a{}_{bcd|e}$ and $g$, (2) holds and (3.1) fails on $V$.

Several cases of these results can now be considered which cover each of the curvature classes $A$, $B$, $C$ and $D$ and for none of which $\Gamma'$ is locally metric.

Consider first the situation where the original $g$ is a vacuum metric of the Goldberg–Kerr type [21, 22]. Then the associated spacetime may be taken as $V$ and admitting a recurrent null vector field $l$ as described above. The Petrov type is III and so the curvature tensor (being equal to the Weyl tensor because of the vacuum condition) has an associated map $f$ of rank 4 (see, e.g., [4]). The curvature class is thus $A$. If $\Gamma'$ defined as above was compatible with a metric $g'$ on some open subset $W$ of $V$, then $\Gamma$ and $\Gamma'$ are each metric connections on $W$ with the same curvature tensor. It then follows from [4, 5] that, on $W$, $g' = cg (0 \neq c \in \mathrm{R})$ and hence the contradiction that $\Gamma' = \Gamma$ on $W$.

Now consider the situation when $V$ is a spacetime of curvature class $B$ and hence of holonomy type $R_7$ [4, 23] and thus the metric $g$ is of the Bertotti–Robinson type (see, e.g., [24]). Further, if $V$ is chosen to be simply connected (as it always can), then two independent recurrent null vector fields $l$ and $n$ are admitted by $V$ with each having all of the properties that $l$ had in the previous paragraph and satisfying $l^a n_a = 1$, $l_{a;b} = l_a p_b$ and $n_{a;b} = -n_a p_b$ [4, 16, 23]. Thus the tensor $h$ on $V$ defined by $h_{ab} = 2l_{(a}n_{b)}$ is covariantly constant on $V$, $h_{ab;c} = 0$. Now define a symmetric connection $\Gamma'$ by the components $\Gamma'^a{}_{bc} = \Gamma^a{}_{bc} + l'^a l'_b l'_c$ and suppose that $\Gamma'$ is compatible with a metric $g'$ on some open subset $W$ of $V$. Then $\Gamma$

and $\Gamma'$ give the same curvature tensor as before and it follows from [4, 15, 16] (see (10*b*)) that $g' = \phi g + \psi h$ for functions $\phi, \psi : W \to R$. Imposing the condition $g_{ab;c} = 0 = g'_{ab|c}$, recalling that $h_{ab;c} = 0$ and taking $P^a{}_{bc} = l'^a l'_b l'_c$ as before gives

$$\begin{aligned} 0 = g'_{ab|c} &= \phi_c g_{ab} + \phi g_{ab|c} + \psi_{,c} h_{ab} + \psi h_{ab|c} \\ &= \phi_{,c} g_{ab} + \phi\left(g_{ab;c} - g_{ae} P^e{}_{bc} - g_{eb} P^e{}_{ac}\right) + \psi_{,c} h_{ab} + \psi\left(h_{ab;c} - h_{ae} P^e{}_{bc} - h_{eb} P^e_{ac}\right) \\ &\Rightarrow (\phi + \psi)_{,c} g_{ab} = 2(\phi + \psi) l'_a l'_b l'_c \end{aligned} \tag{60}$$

A rank argument shows that $\phi = -\psi$ and hence that $g' = \phi(g - h)$. But then $g'_{ab} l^b = g'_{ab} n^b = 0$ and so $g'$ is degenerate. This contradiction completes the proof.

The final two cases may be dealt with briefly. First, let $(M, g)$ be a spacetime of holonomy type $R_6$ ([4, 16, 23]—and for existence and examples see [25]). Then, restricting to an open subset $U$ of $M$, if necessary, one may assume that, on $U$, there are vector fields $y$ and $l$ with $y$ spacelike and covariantly constant and $l$ null and recurrent. Thus $y_{a;b} = 0$ and $l_{a;b} = l_a p_b$ with $p$ a nowhere zero 1-form on $U$ and one may assume that $U$ is chosen so that the map $f$ (section 2) has everywhere rank 2. Then, on $U$, $R^a{}_{bcd} y^d = 0$ (that is, the curvature class is $C$) and $R^a{}_{bcd} l^d$ is nowhere zero. On scaling $l$ to $l'$ as in the previous examples and constructing the connection $\Gamma'$ as before, one again achieves (58) and (59) so that $g$, $\Gamma'$ and $R'$ satisfy (2) but not (3.1). Because of (58) it follows from (10*c*) [4, 5, 15] that if $g'$ is a metric compatible with $\Gamma'$, one must have $g'_{ab} = \phi g_{ab} + \psi y_a y_b$ for functions $\phi, \psi : U \to \mathbf{R}$. Now the condition $g'_{ab|c} = 0$ yields, by an argument which is very similar to (60), the consequence that $\phi \equiv 0$ on $U$ and so that such a tensor $g'$ is not non-degenerate. Hence $\Gamma'$ is not locally a metric connection.

Finally for a spacetime of holonomy type $R_2$ [4, 16, 23, 25] one may arrange an open subset $U$ of it to admit vector fields $l$, $n$, $x$ and $y$ with all inner products between them vanishing on $U$ except $l^a n_a = x^a x_a = y^a y_a = 1$. Also, on $U$, $l$ and $n$ are (null and) recurrent and $x$ and $y$ are covariantly constant. The set $U$ may be chosen so that the map $f$ has rank 1 everywhere on $U$ (and hence, the curvature class is $D$). One then proceeds as above to construct $\Gamma'$ (using $l$ or $n$) so that $g$, $\Gamma'$ and $R'$ satisfy (2) but not (3.1) and notes that, from (58), any metric $g'$ compatible with $\Gamma'$ satisfies [4, 5, 15] $g'_{ab} = \phi g_{ab} + \alpha x_a x_b + \beta y_a y_b + 2\gamma x_{(a} y_{b)}$ for functions $\phi, \alpha, \beta, \gamma : U \to \mathbf{R}$. Another argument, very similar to (60), then shows that $\phi \equiv 0$ on $U$ and thus $g'$ is not non-degenerate. Hence $\Gamma'$ is not locally a metric connection.

As a final remark, it is noted how the tensor $h$ and any local metric compatible with the connection $\Gamma$ are related geometrically. For the case relevant to theorem 2(i), that is, class $A$, any local metric is just a conformal rescaling of $h$. For class $B$, any local metric is a (functional) combination of $h$ and the local covariantly constant tensor $t$, where the latter tensor's eigenspaces (with respect to $h$) are, at each relevant point $m$, identical to the two two-dimensional subspaces of $T_m M$ into which $T_m M$ is decomposed by the curvature tensor for this class. For class $C$, any local metric is either a rescaling of $h$ or a (functional) combination of $h$ and the covariantly constant tensor $k_a k_b$ where $k$ is a local covariantly constant vector field spanning the unique direction picked out at each relevant point by a class $C$ curvature tensor. For class $D$, any local metric $g$ is either a rescaling of $h$ or a (functional) combination of $h$ and appropriate products of either one or two local covariantly constant vector fields and where these latter vector fields lie in the two-dimensional subspace determined at each appropriate point by this curvature class. In addition, it is noted that if $g$ and $g'$ are two local metrics compatible with $\Gamma$ in some neighbourhood $U$, the latter being contained in one of the regions $A$, int $B_1$, int $B_2$, etc considered above, then $g$ and $g'$, having the same Levi-Civita connection, are either conformally related to a constant conformal factor or are jointly related to the appropriate covariantly constant tensor(s) mentioned above by simple (constant coefficient) combinations and which reflect the local holonomy group [3, 4].

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